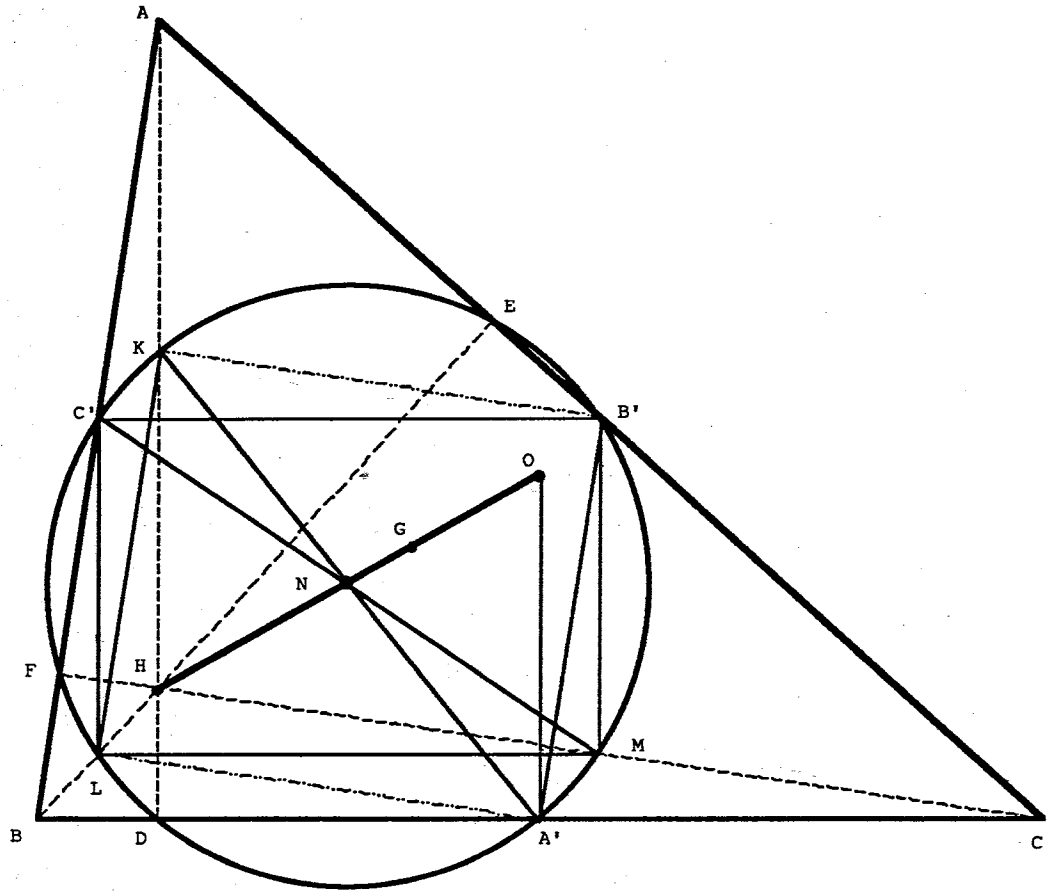
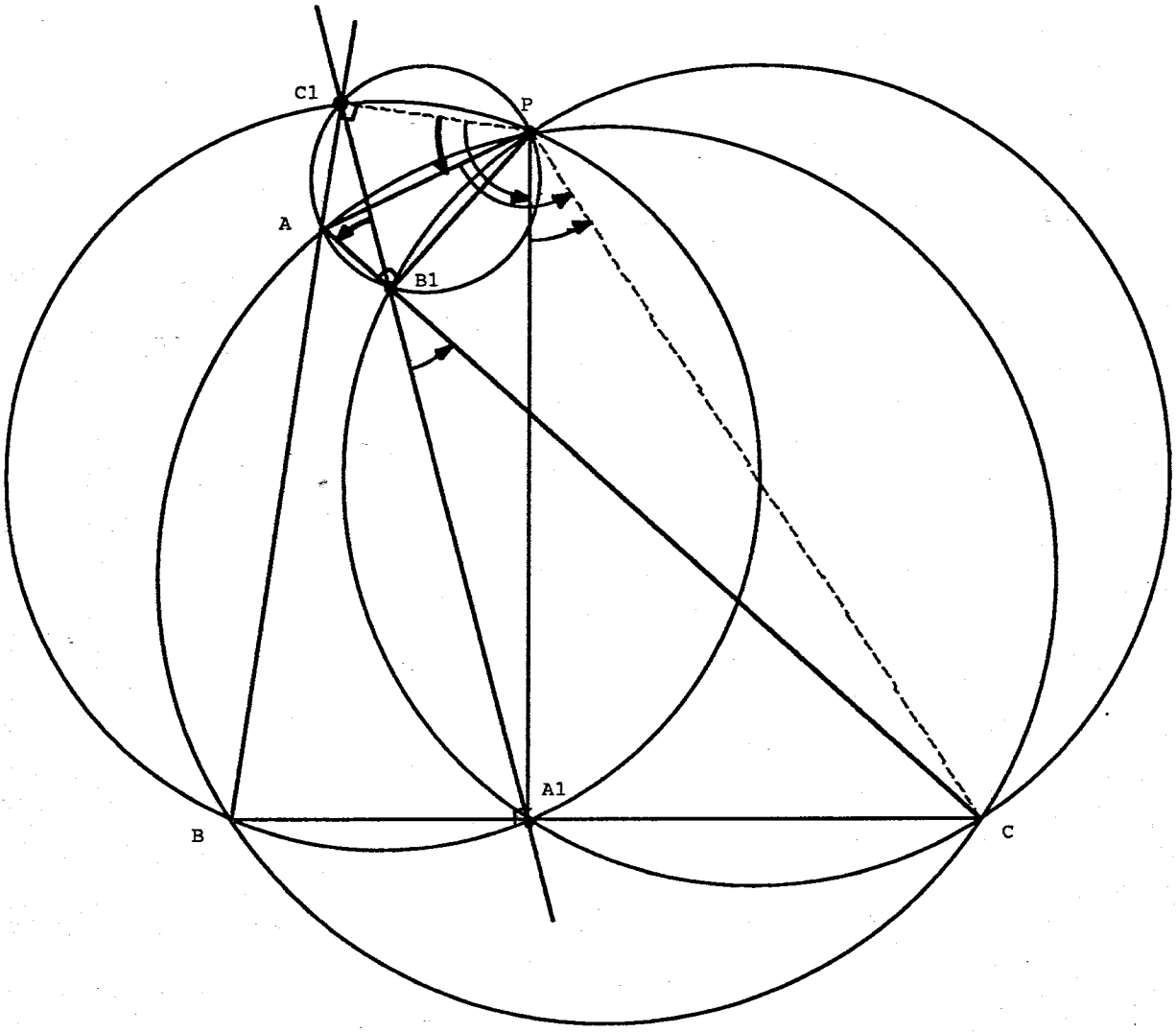
Geometry III - Summer 1992 The nine-point circle



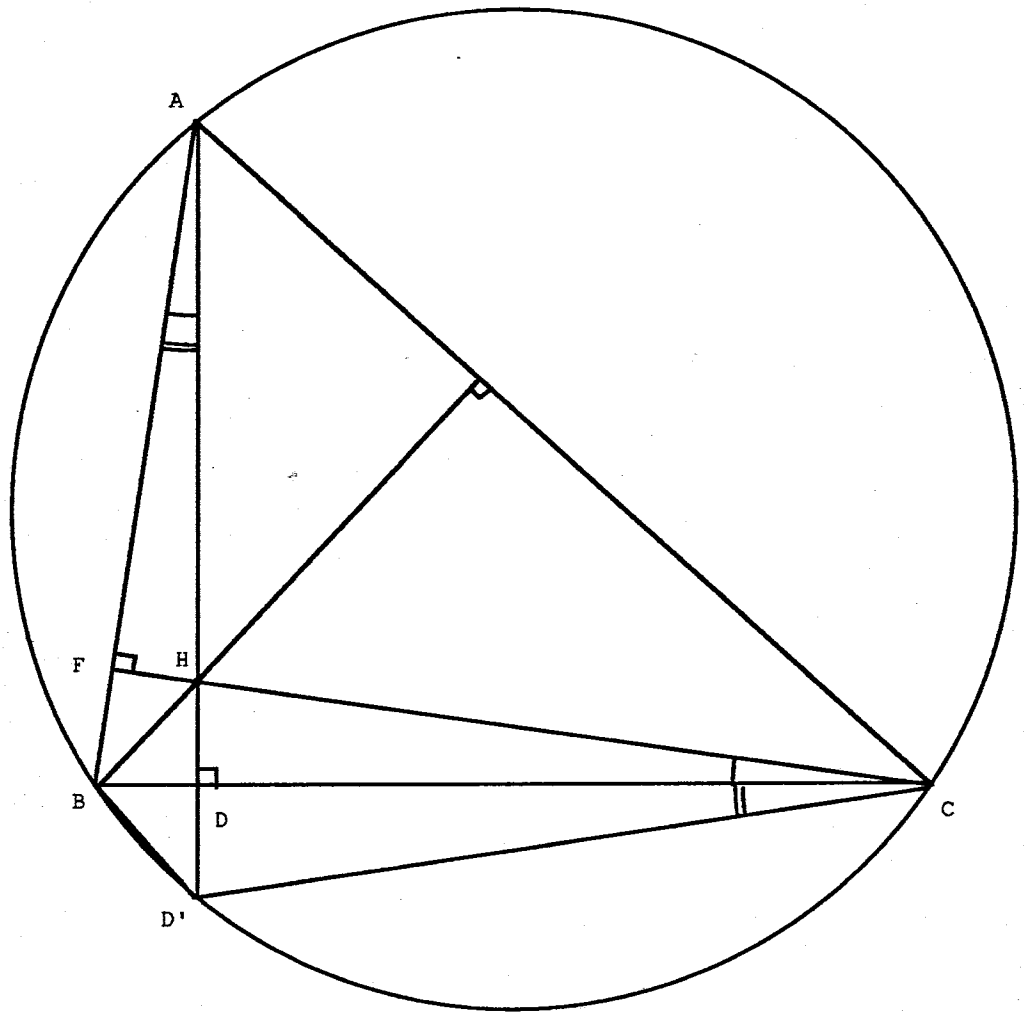
Theorem The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocenter, all lie on the same circle. The radius of this circle is half that of the circumcircle.

Summary of the key ideas in the steps of the proof.

- C' and B' are the midpoints of AB and AC , respectively, so C'B' is parallel to BC and half as long. Similarly, LM is parallel to BC and half as long. Hence C'B'M L is a parallelogram.
- C' and L are the midpoints of BA and BH , respectively, so C'L is parallel to AH . AH is perpendicular to BC , so C'L is perpendicular to LM . Hence, C'B'M L is a rectangle. Likewise, C'A'M K is a rectangle. These two rectangles each have C'M as a diagonal.
- Let N be the midpoint of C'M . The circle with center N and passing through C' and M passes through B' since M B'C' is a right triangle with hypotenuse a diameter of the circle. Similarly, this circle passes through L, A', K .
- A'K is a diameter of the circle and KD is perpendicular to DA' , so D lies on the circle. Likewise, E and F lie on this circle.
- Triangle KLM is congruent to triangle A'B'C' (SSS), H is the orthocenter of triangle KLM , O is the orthocenter of triangle A'B'C' , so KH = A'O . Since N is a midpoint, KN = A'N . Angle HKN = angle OA'N by KH parallel to A'O , so triangle HKN is congruent to triangle OA'N (SAS) . Thus, angle HNK = angle ONA' , so N lies on the line HGO and HN = NO .



A B C P and C_1 B A_1 P are concyclic, so $\angle$ APC = 180° - $\angle$ B = $\angle$ C_1PA_1
Subtract $\angle$ APA_1 to obtain $\angle$ C_1PA = $\angle$ A_1PC (oriented angles).
B_1 A_1 C P are concyclic, so $\angle$ A_1PC = $\angle$ A_1B_1C .
C_1 A B_1 P are concyclic, so $\angle$ C_1PA = $\angle$ C_1B_1A .
Hence, oriented angles $\angle$ A_1B_1C = $\angle$ C_1B_1A and A_1 B_1 C_1 are collinear.

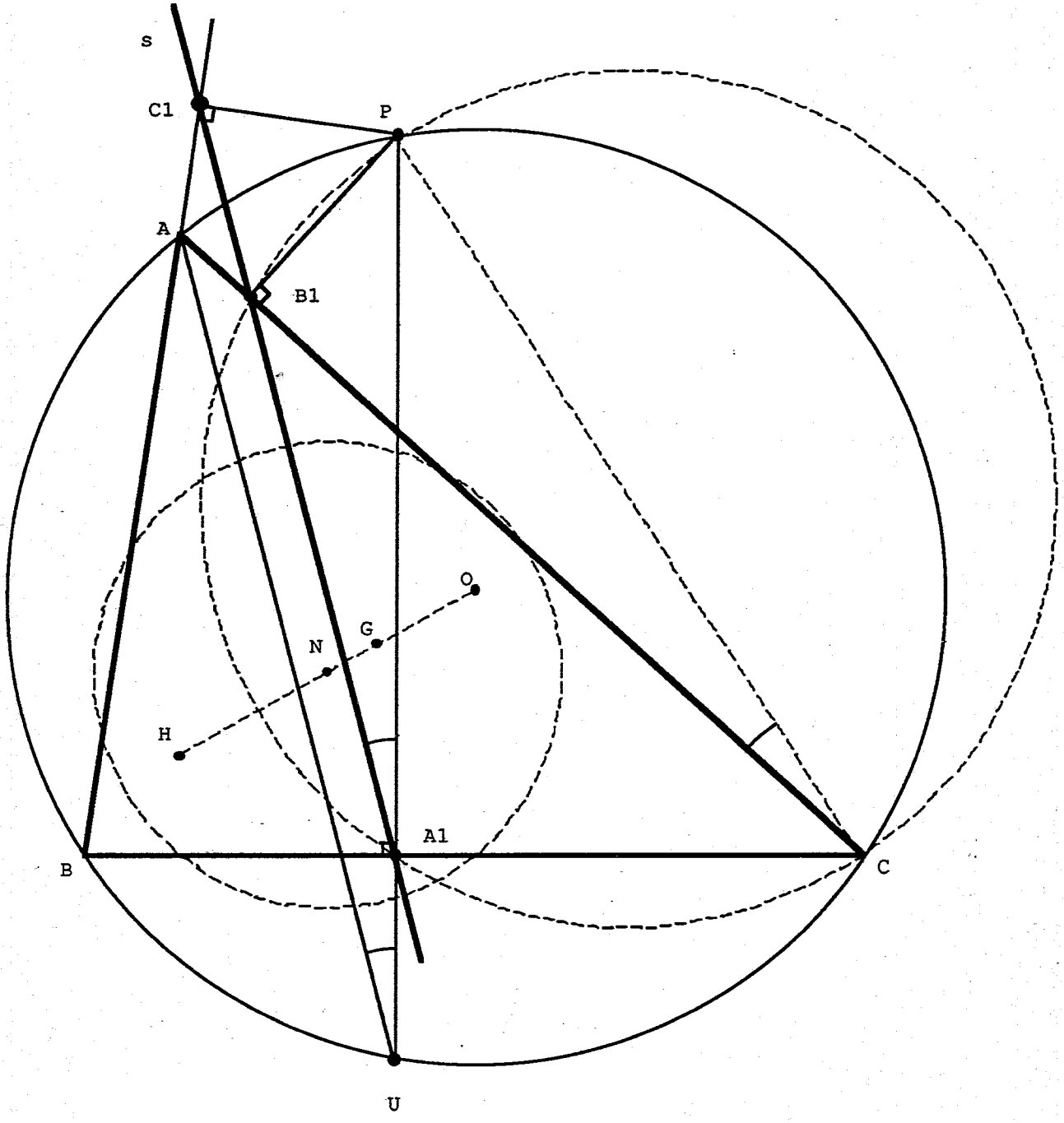


Extend the altitude AD to meet the circumcircle at D' .

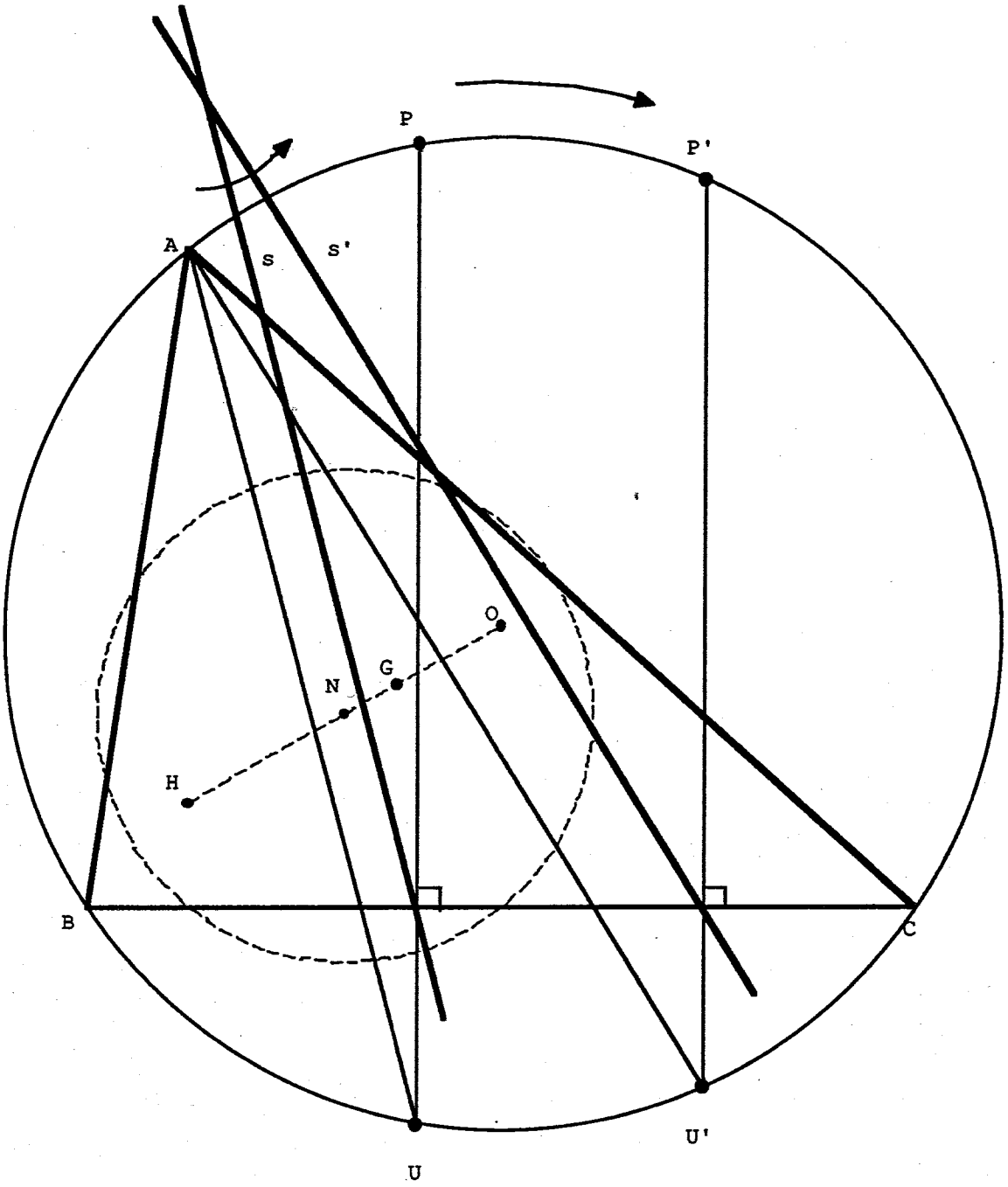
Claim: HD = DD' .

For: $\angle$ FCB = 180° - $\angle$ B = $\angle$ BAD and $\angle$ BAD' = $\frac{1}{2}$ arc(BD')

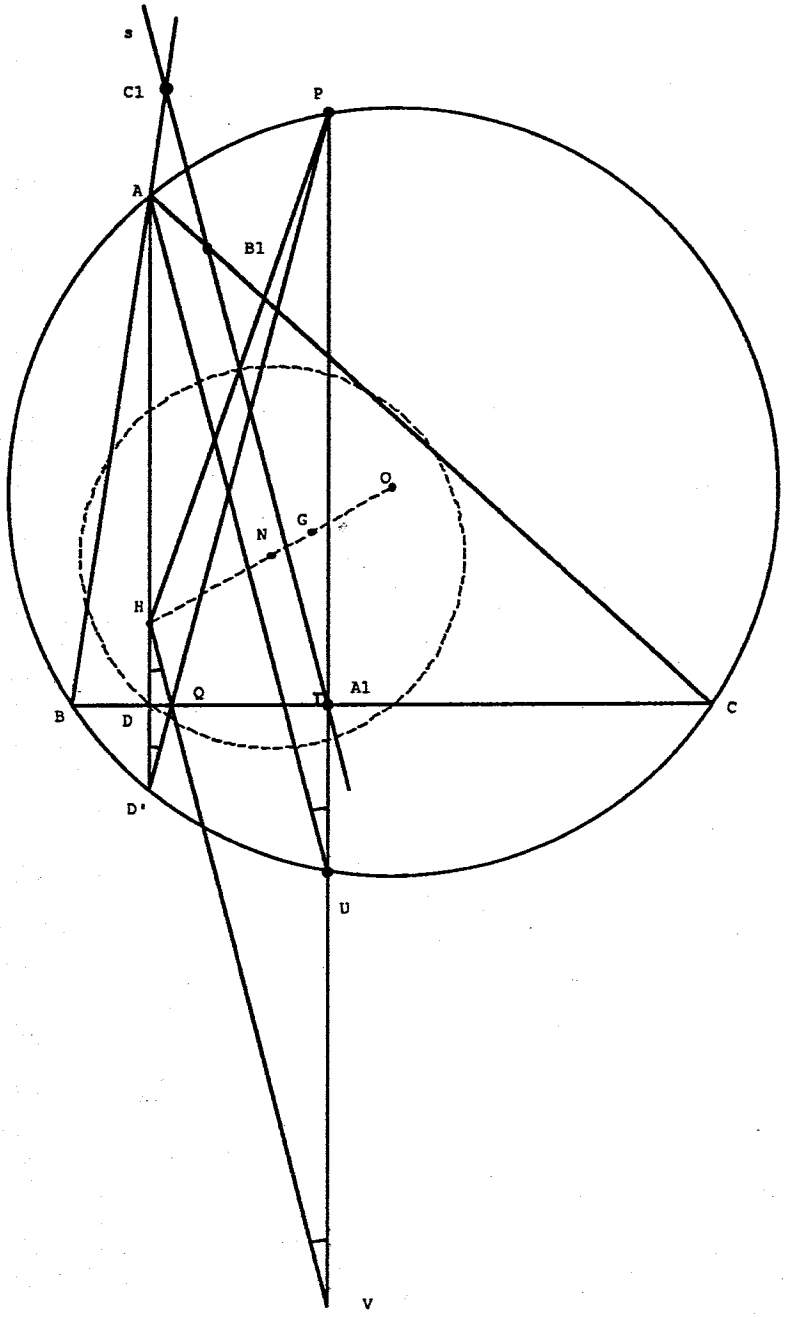
= $\angle$ BCD' , so $\angle$ HCD = $\angle$ D'CD and rt Δ HDC $\cong$ rt Δ D'DC .



• Extend PA_1 to U on the circumcircle. Claim: $AU \parallel B_1A_1$.
For: $P\ A\ U\ C$ are concyclic, and $P\ B_1\ A_1\ C$ are concyclic.
The two circles share diameter PC, so
$\angle\ B_1A_1P = \frac{1}{2}\ \text{arc}(B_1P) = \angle\ ACP = \frac{1}{2}\ \text{arc}(AP) = \angle\ AUP$.
Thus, PU is a transversal making the same angles with AU and B_1A_1.



• Let s and s' be the Simson lines of P and P'. Now $AU \parallel s$ and $AU' \parallel s'$, so the angle between the Simson lines s and s' is $\angle UAU' = \frac{1}{2}\,\mathrm{arc}(UU') = \frac{1}{2}\angle UOU' = -\frac{1}{2}\angle POP'$. The "motion" of Simson line s of P is retrograde to that of the point P on the circumcircle.



Claim: The Simson line s of the point P bisects HP .

• Let Q be the point where D'P meets BC . Let V be the point where the perpendicular PA_1 from P to BC meets HQ (extended).

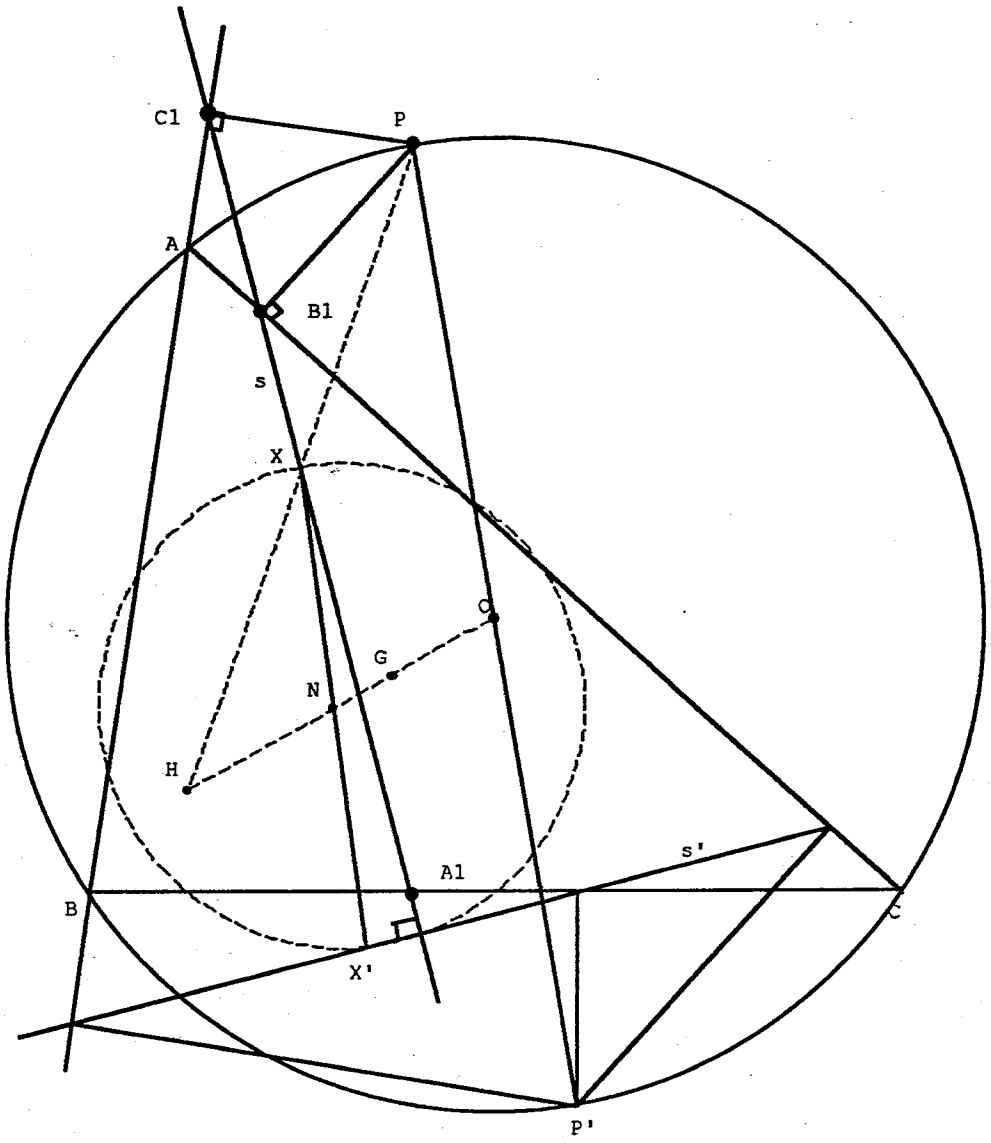
• Since HD = DD' , $\text{rt}\,\Delta\, D'DQ \cong \text{rt}\,\Delta\, HDQ$. Thus, $\angle\, DD'Q = \angle\, DHQ = \angle\, HVP$.

• D' U P A are concyclic, so $\angle\, AD'P = \angle\, AUP$.

Hence, $\angle\, DHQ = \angle\, DD'Q = \angle\, AUP = \angle\, HVP$, and $HV \parallel AU$.

• $\text{rt}\,\Delta\, QA_1V \cong \text{rt}\,\Delta\, QA_1P$, so A_1 bisects PV .

• In $\Delta\, HVP$, s is parallel to side HV , and bisects side PV (at A_1), so it bisects side PH also.



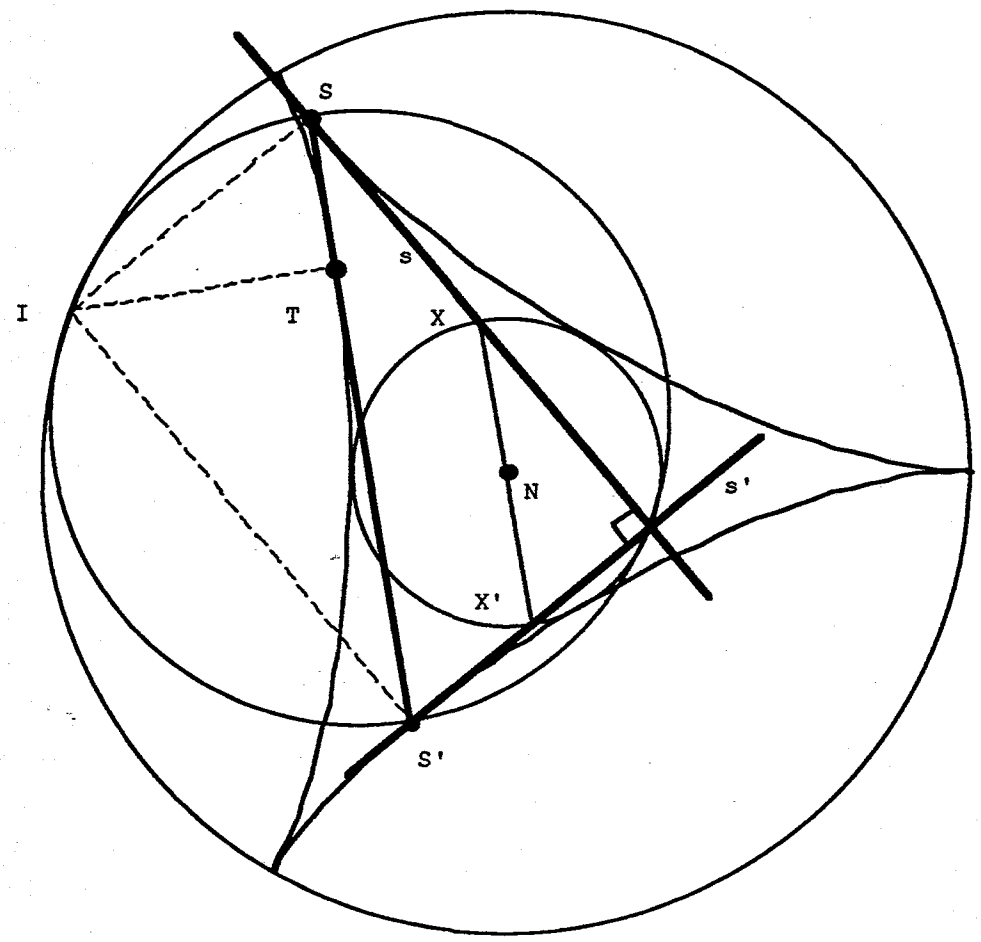
Claim: The Simson lines s and s' of diametrically opposite points P and P' meet at right angles on the nine-point circle.

• Let X be the point where s meets HP. X is the midpoint of HP, N is the midpoint of HO, $\Delta NHX \sim \Delta OHP$, so $NX = \frac{1}{2} OP = \frac{1}{2} R$. Hence X lies on the nine-point circle.

• Similarly, s' meets HP' at X' on the nine-point circle, XX' is parallel to PP', and XX' passes through N.

• Since $\text{arc}(PP') = 180°$, s and s' meet at right angles. The hypotenuse of the right triangle with sides s, s', and XX' is a diameter of the nine-point circle, so its right angle lies on that circle.

• Remark: The dilation with center H and factor $1/2$ sends the point P above to X.



• Let a circle of radius 2a roll without slipping inside a fixed circle of radius 3a and center N ; let SS' be a diameter affixed to the rolling circle. The locus of either endpoint S or S' is a hypocycloid of three cusps (or a "deltoid"); the diameter SS' is tangent to the hypocycloid at T , so the curve is the "envelope" of this diameter.
• The point I at which the rolling circle is tangent to the circle of radius 3a is the "instantaneous line of centers" of the rotation. The lines IS , IS' , and IT are at right angles to the tangents s , s' , and SS' to the hypocycloid at S , S' , and T . The tangents s and s' are perpendicular and meet at a point at distance a from N .
• If the circle with center N and radius a is the nine-point circle of a triangle, the lines s and s' are Simson lines of the triangle and the hypocycloid is the envelope of the Simson lines.

References

E.H. Lockwood, A Book of Curves, Cambridge U. Press, 1963. Chapter 8.
H.S.M. Coxeter and S.L Greitzer, Geometry Revisited, NML 19, The Mathematical Association of America, Washington, D.C., 1967. Section 2.5 ff.